Problem Set 3 due March 11, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

(1) Given numbers a and b, for which number c does the system:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

have a solution v_1, v_2 (you must express c in terms of a and b). *(10 points)*

(2) Draw the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfing the conditions in part (1) on a picture of $\mathbb{R}^3$. *(5 points)*

(3) Construct a 3×3 matrix whose column space is spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. *(5 points)*

Problem 2:

Consider the following planes in 3–dimensional space:

- the xz coordinate plane
- the plane cut out by the equation $2x + 3y = 5z$

(1) Write down a 3×3 matrix A whose nullspace is the intersection of the two planes above.

(10 points)

(2) Give examples of vectors $\boldsymbol{b} \in \mathbb{R}^3$ for which the system of equations $A\boldsymbol{v} = \boldsymbol{b}$ has:

- no solutions *(5 points)*
- infinitely many solutions *(5 points)*

Problem 3:

(1) Compute the reduced row echelon form of the matrix:

$$A = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & 3 & -5 & -1 \end{bmatrix}$$

(all zero rows should be at the bottom of A). *(10 points)*

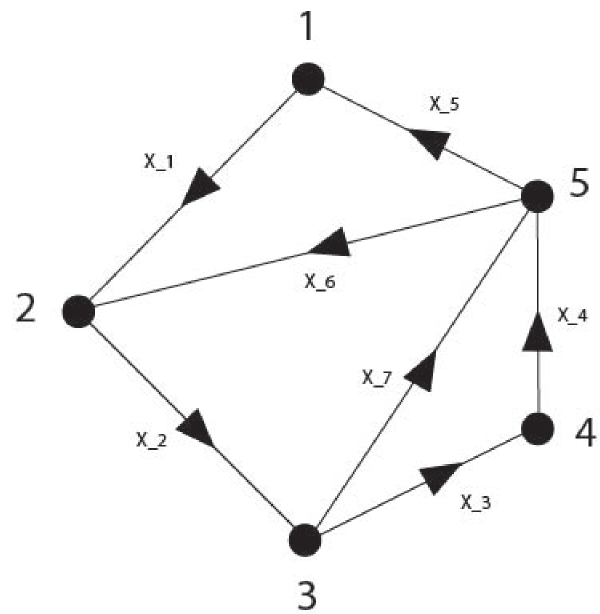
(2) Use the result of part (1) to find the full set of solutions to the equation:

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

(10 points)

Problem 4:

The diagram below represents 5 nodes (represented by the circles) connected by 7 pieces of conducting wire (represented by the lines). The intensity of the electric current flowing through these pieces of wire is $x_1, ..., x_7$, in the direction of the arrow. If any of the x_i's are negative, this just means current flowing in opposite direction to the arrow.



Kirchoff's first law says that, at every node, the incoming current should equal the outgoing current.

(1) Write down explicitly the incidence matrix of the diagram. By definition, this is the 5×7 matrix A whose entry at row i and column j is:

$$\begin{cases} 1 & \text{if the current on the } j\text{–th wire flows into node } i \\ -1 & \text{if the current on the } j\text{–th wire flows out of node } i \\ 0 & \text{if the } j\text{–th wire does not intersect node } i \end{cases}$$

(the j–th wire is the one denoted by the variable x_j in the diagram). *(5 points)*

(2) Express Kirchoff's first law as a linear algebra condition on the vector of currents $\begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$, which involves the incidence matrix A (justify). *(5 points)*

(3) By using the reduced row echelon form of A, find all possible vectors of currents $\begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$ which satisfy Kirchoff's law for the diagram above. *(10 points)*

Problem 5: (justify all your answers)

(1) If X is an invertible square matrix, what can you say about $C(X)$ and $N(X)$? *(10 points)*

(2) If $Y = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$ is a block matrix, what is $N(Y)$ in terms of $N(A)$ and $N(B)$? *(5 points)*

(3) If $Z = \left[\begin{array}{c|c} A & B \end{array}\right]$ is a block matrix, what is $C(Z)$ in terms of $C(A)$ and $C(B)$? *(5 points)*